Exercise 27

If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point $(1, 2)$.

Solution

Determine the derivative of $f(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[3(x+h)^2 - (x+h)^3] - [3x^2 - x^3]}{h} \\
&= \lim_{h\to 0} \frac{[3(x^2 + 2xh + h^2) - (x^3 + 3x^2h + 3xh^2 + h^3)] - 3x^2 + x^3}{h} \\
&= \lim_{h\to 0} \frac{(3x^2 + 6xh + 3h^2 - x^3 - 3x^2h - 3xh^2 - h^3) - 3x^2 + x^3}{h} \\
&= \lim_{h\to 0} \frac{6xh + 3h^2 - 3x^2h - 3xh^2 - h^3}{h} \\
&= \lim_{h\to 0} (6x + 3h - 3x^2 - 3xh - h^2) \\
&= 6x - 3x^2
\end{aligned}$$

Plug in $x = 1$ to this formula to get $f'(1)$.

$$f'(1) = 6(1) - 3(1)^2 = 6 - 3 = 3$$

This is the slope of the tangent line to the curve at $x = 1$. Use the point-slope formula and the provided point $(1, 2)$ to get the equation of this line.

$$y - 2 = 3(x - 1)$$

$$y - 2 = 3x - 3$$

$$y = 3x - 1$$

Below is a graph of the curve along with the tangent line at $x = 1$.